**Queueing Syst manuscript No.**
(will be inserted by the editor)

# Stability of longest-queue-first scheduling in linear wireless networks with multihop traffic and one-hop interference

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Received: date / Accepted: date

**Abstract** We consider the stability of the longest-queue-first (LQF) scheduling policy in wireless networks with multihop traffic under the one-hop interference model. Although it is well known that the back-pressure algorithm achieves the maximal stability, its computational complexity is prohibitively high. In this paper, we consider LQF, a low-complexity scheduling algorithm, which has been shown to have near-optimal throughput performance in many networks with single-hop traffic flows. We are interested in the performance of LQF for multihop traffic flows. In this scenario, the coupling between queues due to multihop traffic flows makes the local-pooling-factor analysis difficult to perform. Using the fluid-limit techniques, we show that LQF achieves the maximal stability for linear networks with multihop traffic and a single destination on the boundary of the network under the one-hop interference model.

**Keywords** Queueing networks · Stability · Longest-queue-first scheduling · Linear networks · Fluid limit · Throughput optimality · Multihop traffic

An earlier version of the paper appeared in IEEE CDC 2013 [12].
This work was supported in part by the National Science Foundation (NSF) under Grants CNS-1264012 and CNS-1262329, and in part by the Defense Threat Reduction Agency (DTRA) under Grant HDTRA1-13-1-0030.

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## 1 Introduction

The scheduling problem in wireless networks with multihop traffic has gained significant attention over the last few decades. One fundamental goal of the design of scheduling policies, among many others, is to decide which set of links to schedule at each time slot in accordance with the underlying interference model so that the system is stable. The back-pressure algorithm has been proved to be throughput-optimal for general multihop-traffic settings [19]; i.e., it stabilizes the network as long as the arrival rates are within the network throughput region. The algorithm, however, requires the network to solve a maximum-weight independent set problem at each time instance and requires the nodes to exchange queue lengths with their neighbors constantly.

In this paper, we study the stability of the longest-queue-first (LQF) scheduling policy, which selects links according to the queue lengths in a greedy fashion. LQF has been extensively studied as a low-complexity approximation of MaxWeight scheduling, and has great throughput and delay performance in many networks. The conditions under which LQF is throughput-optimal have been established by Dimakis and Walrand [8] and the performance guarantee of LQF in general networks has been characterized by Joo et al. [11] and estimated under different scenarios [1,10,11,13]. An asynchronous version of LQF has also been proved to be throughput-optimal under the local-pooling condition by Maguluri et al. [14] However, these results all assume single-hop traffic flows in the network. For networks with multihop traffic, transmitted packets at one link may become the *internal* arrivals to another link. Hence links with small queues may affect the ones with large queues by providing internal arrivals, which makes it difficult to analyze the system using local-pooling-factor analysis since the maximum fluid among the large queues does not always decrease, as we will show in the example in Section 3.2. Although Brzezinski et al. [2] developed conditions for networks with multihop traffic under which a back-pressure-based greedy algorithm achieves the maximal throughput, the performance of LQF for networks with multihop traffic flows is still unknown. We are interested in tackling the problem of throughput performance of LQF, since it can shed light on the implementation of low-complexity scheduling algorithms in wireless multihop networks. While the original LQF is a centralized scheduling policy, a queue-length-based CSMA-type algorithm, called D-GMS, has been proposed by Ni et al. [15] to approximate LQF in a distributed fashion, which does not require constant exchange of queue lengths. Thus, LQF can be used as the foundation to a low-complexity, distributed scheduling algorithm.

We focus on the scheduling problem under multihop traffic in a simple network, i.e., a linear network with single destination on the boundary of the network (also known as a tandem network) and one-hop interference model (also known as primary or node-exclusive interference model) [1]. Such networks

[1] While the one-hop interference model is indeed a mathematical simplification of wireless interference in reality, it has been used as a reasonable approximation to Bluetooth or FH-CDMA networks [11]. It may also be used to model half-duplex communication.

have been well studied in the literature to provide insights in understanding the fundamental scaling properties of multihop traffic [3, 9, 18, 20]. In particular, Stolyar [18] and Bui et al. [3] analyzed the asymptotic delay performance of the back-pressure algorithm in large linear networks when no interference is present. To the best of our knowledge, however, neither throughput nor delay performance guarantee of LQF has been obtained under multihop traffic scenario for linear networks.

This paper proves the throughput optimality of LQF in a special case of linear networks. While the result is only for linear networks, it is the first step to understand the following question: to achieve throughput optimality in a wireless network with multihop traffic flows that have fixed routes, is it sufficient to use queue lengths as weights instead of using differential queues? If the answer is positive, then nodes do not need to constantly exchange queue lengths, which eliminates a significant amount of communication overhead.

The novelty in this paper lies in the techniques we adopt to show the stability of the fluid model after the standard construction of fluid limits. Instead of using an explicit Lyapunov function, we follow the observations from the simulation trajectories of an example network and examine the evolution of the states of the deterministic fluid limits. We first show that the system will eventually stay in the state where the fluid at the first queue is zero. Then by combining the first two queues into one using a coupled-network argument, we reduce the size of the network by one and conclude that fluids at all queues eventually become zero by induction. [2]

The paper is organized as follows. We introduce the basic model in Section 2. In Section 3 we present our result of throughput optimality of LQF, as well as an intuitive example, formal notation and network equations, construction of fluid limits, and the proof. Section 4 concludes the paper.

## 2 Model

Consider a linear network represented by a directed graph $G = (V, L)$ with $|V| = N + 1$ nodes and $|L| = N$ links as shown in Fig. 1. Let $V = \{v_1, v_2, \ldots, v_{N+1}\}$ and $L = \{l_1, l_2, \ldots, l_N\}$, where $l_i$ is the link from node $v_i$ to node $v_{i+1}$. We assume $v_i$ is the origin node of flow $f_i$ with exogenous (or external) packet arrival rate $\alpha_i$ for $1 \le i \le N$, and all flows have the same destination $v_{N+1}$. In the paper we focus on the one-hop interference model, so the interference graph is as shown in Fig. 2.

We assume time is slotted, and in each time slot a subset of the links can be scheduled. Once scheduled, a packet at link $l_i$ is transmitted from node $v_i$ to node $v_{i+1}$ and join the queue at node $v_{i+1}$ if it has not reached the destination $v_{N+1}$, or leave the network otherwise. As a result, besides exogenous packet arrivals, there can also be *internal* packet arrivals to a node according to the schedule of other links.

[2] We remark that while a properly chosen Lyapunov function would suffice to show stability, finding such a function may be difficult.

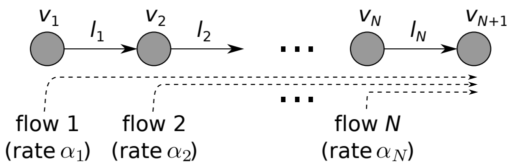


**Fig. 1** A linear network with $N$ links. The $i$th dashed line indicates the flow with source node $v_i$ and destination node $v_{N+1}$ and exogenous packet arrival rate $\alpha_i$.

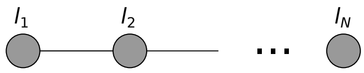


**Fig. 2** The one-hop interference graph of Fig. 1.

The scheduler decides a subset of the links $s \subseteq L$ to be activated in every time slot, called a schedule, such that the schedule is feasible (scheduled links do not interfere with each other) and maximal (no other link can be added to the schedule without violating the feasibility constraint), and then the queue length at each transmitter in the activated subset reduces by 1 if there are any packets to schedule, or remains 0 otherwise. A schedule (also known as an activation set) $s$ is represented by an *activation vector* $m$, which is a binary column vector with $N$ elements. According to the interference model shown in Fig. 2, a schedule $s$ is feasible if no two adjacent links are activated at the same time; i.e., the activation vector $m$ does not contain two consecutive 1's. For example, the activation vectors for the four maximal schedules when $N = 5$ are 10101, 10010, 01010 and 01001. The number of maximal schedules grows exponentially with $N$.

In the paper we are interested in LQF with arbitrary tie-breaking rules, and we define it as follows. At each time slot, let $Z_i$ be the queue length at link $l_i$ for $1 \le i \le N$. The set of links are sorted with arbitrary tie-breaks such that $Z_{\sigma_1} \ge Z_{\sigma_2} \ge \cdots \ge Z_{\sigma_N}$, where $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ is the sorted index vector. LQF starts with the schedule $\mathcal{E} = \{\sigma_1\}$, and proceeds to consider $i = 2, 3, \ldots, N$ inductively and appends $\sigma_i$ to $\mathcal{E}$ if $\sigma_i$ does not interfere with any link that is already in $\mathcal{E}$. This procedure ends after the link $l_{\sigma_N}$ is considered and the resulting schedule $\mathcal{E}$ is the schedule chosen by LQF.

## 3 Stability

In this section we analyze the stability property of LQF in the linear network under the one-hop interference model. We say the system is *stable* if the queue length process, as a Markov process, is positive recurrent. We first state the main theorem with the proof outline and an illustrative example, and then proceed with the formal proof.

### 3.1 Main Result

**Theorem 1** *LQF is throughput-optimal on linear networks with multihop traffic and a single destination on the boundary of the network under the one-hop interference model.* ◇

Theorem 1 states that LQF can stabilize a linear multihop traffic network with a single destination on the boundary and the one-hop interference model. Thus using queue lengths instead of queue differences is sufficient. This result may also shed light on the throughput performance of LQF in other networks with multihop traffic, in which the routes are fixed.

The proof consists of the following steps. We first follow the standard construction of the fluid limits. Then we show that eventually the fluids should be such that each fluid is less than or equal to at least one neighbor fluid; i.e., no fluid dominates all its neighbors. After that we prove that the first fluid must decrease with rate at least $\epsilon > 0$. Finally we use a coupled-network argument to show that all fluids eventually go to zero under admissible arrival rates, which implies throughput optimality.

We next demonstrate the key ideas of the proof using an example.

### 3.2 Three-Link Linear Network

We consider the simple linear network example with four nodes $\{v_1, v_2, v_3, v_4\}$ and three links $\{l_1, l_2, l_3\}$. Suppose flow $i$ has origin $v_i$ and destination $v_4$ with Bernoulli arrival of rate $\alpha_i$ for $i = 1, 2, 3$. The interference is such that two adjacent links cannot be scheduled at the same time, so either $\{l_1, l_3\}$ or $\{l_2\}$ is scheduled in each time slot. Let $Z_i(n)$ be the queue length on link $l_i$ at time slot $n$. Then at each time slot, the LQF scheduler first selects the longest queue with arbitrary tie-breaking, and then chooses either $\{l_1, l_3\}$ or $\{l_2\}$ according to the first selection.

A typical queue evolution graph for the three-link linear network under LQF is shown in Fig. 3. Here the initial queue lengths are $Z_1(0) = 300$, $Z_2(0) = 120$ and $Z_3(0) = 100$, with arrival rates $\alpha_1 = 0.25$, $\alpha_2 = 0.1$ and $\alpha_3 = 0.05$. We make several interesting observations from the figure:

1. The queue lengths look like piecewise-linear functions (this is partially due to the law of large numbers over the arrival process).
2. The queue dynamics are somewhat complex for the first time slots (largely due to the internal arrival from other links).
3. The first queue eventually drops to close to zero, and the behavior of the other queues become more predictable.
4. Finally all queues seem to be close to zero, so the system is expected to be stable.

In light of the above findings, we first conjecture that after some time we have either $Z_1(t) \approx Z_2(t) \geq Z_3(t)$ or $Z_1(t) \leq Z_2(t) \approx Z_3(t)$, since otherwise one queue will be larger than all its neighbors, resulting in a higher scheduling

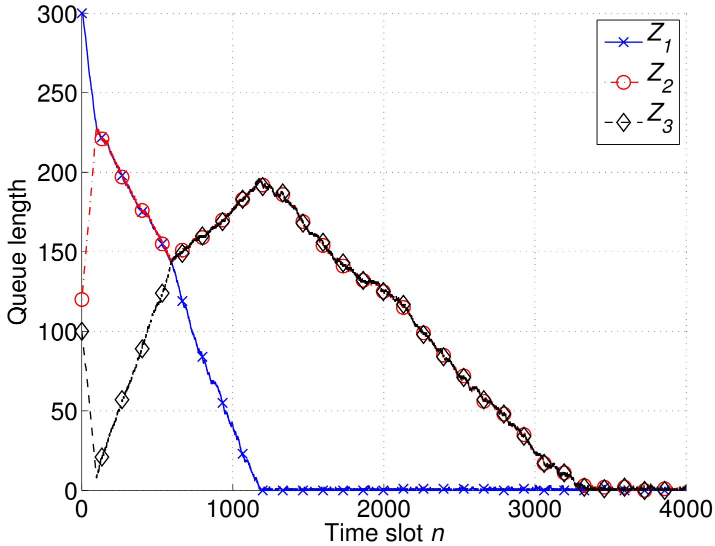


**Fig. 3** Queue evolution of the three-link linear network under LQF.

priority under LQF that will force the queue to start decreasing. We can then see that if $Z_1(t)$ and $Z_2(t)$ stick together then they must both decrease since the sum of the nominal total arrival rates to links $l_1$ and $l_2$ due to both the exogenous and internal arrivals is $2\alpha_1 + \alpha_2 = 0.6 < 1$. If $Z_2(t)$ and $Z_3(t)$ stick together and $Z_1(t)$ is positive, then the service rate of link $l_i$ for all $i$, denoted by $\mu_i$, should satisfy

$$\mu_1 = \mu_3,$$

$$\mu_1 + \mu_2 = 1,$$

and

$$\mu_1 + \alpha_2 - \mu_2 = \mu_2 + \alpha_3 - \mu_3.$$

We can then get $\mu_1 - \alpha_1 = \frac{1}{2} - \alpha_1 - \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 = 0.2375 > 0$. So in either case the first queue will decrease. We also argue that when the first queue drops to close to zero, it cannot rise again since if it did it would be "forced back" to zero immediately since the potential service rate of link $l_1$ is larger than its nominal (exogenous only) arrival rate. So at last the three-link linear network is reduced to a two-link linear network and the remaining two queues go to close to zero as well. The above intuition will lead our way to the rigorous proof for the general linear network case in the rest of this paper.

### 3.3 Notation and Network Equations

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{R}_+$ the set of nonnegative real numbers. We use curly braces like $\{x(n), n \in \mathbb{N}\}$, $\{x_n, n \in \mathbb{N}\}$

or $\{X(n), n \in \mathbb{R}_+\}$ for a sequence or a stochastic process with index $n$, and use the shorthand $\{x(n)\}$, $\{x_n\}$ or $\{X(n)\}$ whenever appropriate. We reserve the curly braces with vertical bars for set-builders. We also use the following notation:

- $R$: the $N$-by-$N$ routing matrix that is similar to the one defined by Tassiulas and Ephremides [19], where $R_{ik} = -1$ if link $l_k$ goes from node $v_i$, $R_{ik} = 1$ if link $l_k$ goes to node $v_i$, or $R_{ik} = 0$ otherwise, for $1 \le i \le N$ and $1 \le k \le N$. Then in the linear network case the routing matrix is given by

$$R = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{1}$$

- $M$: the $N$-by-$r$ binary-entry matrix whose columns are the activation vectors of the possible maximal schedules, where $r$ is the total number of possible maximal schedules. We then have $M = (m_1, m_2, \ldots, m_r)$ where $m_j$ is the activation vector of a maximal schedule for $1 \le j \le r$.
- $Z_i(n)$ for $1 \le i \le N$: the queue length at link $l_i$ at time slot $n \in \mathbb{N}$ (before arrivals and departures happen in time slot $n$).
- $E_i(n)$ for $1 \le i \le N$: the cumulative exogenous arrival to link $l_i$ up to time slot $n \in \mathbb{N}$. We assume the increment process of $\{E_i(n)\}$, i.e., the process $\{E_i(n+1) - E_i(n), n \in \mathbb{N}\}$, is i.i.d. across $n$. We also assume the processes $\{E_1(n)\}$, $\{E_2(n)\}$, $\ldots$, $\{E_N(n)\}$ are independent. The exogenous arrival rate is $\mathbb{E}[E_i(n+1) - E_i(n)] = \alpha_i$ for all $n$.
- $A_i(n)$ for $1 \le i \le N$: the cumulative arrival to link $l_i$ up to time slot $n \in \mathbb{N}$. This includes both exogenous and internal arrivals.
- $D_i(n)$ for $1 \le i \le N$: the actual cumulative departure from link $l_i$ up to time slot $n \in \mathbb{N}$.
- $T_j(n)$ for $1 \le j \le r$: the cumulative service time (in number of time slots) of schedule $m_j$ up to time slot $n \in \mathbb{N}$.
- $Y_i(n)$ for $1 \le i \le N$: the cumulative idle time (in number of time slots) of link $l_i$ up to time slot $n \in \mathbb{N}$ (when link $l_i$ is chosen by the scheduler but does not actually send packets). Note that even if the queue at link $l_i$ is empty at the time of scheduling, the scheduler can still choose the schedule $m_k$ such that $l_i \in m_k$, in which case $Y_i(n)$ will increase instead of $D_i(n)$. For non-idling (or work-conserving) scheduling policies $Y_i(n)$ can only increase when the queue at link $l_i$ is empty.

Let $Z(n), E(n), A(n), D(n), T(n), Y(n)$ be the corresponding column vectors and let $\mathbb{X}(n) = (Z(n), E(n), A(n), D(n), T(n), Y(n))$ for any $n \in \mathbb{N}$. Then we refer to $\{\mathbb{X}(n)\}$ as the *queueing network process*. Let $\mathcal{X} = \mathbb{N}^{5N+r}$ be the state space of $\{\mathbb{X}(n)\}$. Then $\{\mathbb{X}(n)\}$ is an $\mathcal{X}$-valued stochastic process defined on $\mathbb{N}$. Let $\Omega$ be the set of sample paths specifying the exogenous arrival processes $\{E(n)\}$ and the possible tie-breaks of the scheduler. Note that under the

LQF policy $\{\mathbb{X}(n)\}$ is a discrete Markov chain. The dynamics of the network are described by the following *queueing network equations*:

$$A(n) = E(n) + (R + I_N)D(n-1) \tag{2}$$

$$Z(n) = Z(0) + A(n) - D(n) \tag{3}$$

$$\sum_{j=1}^{r} T_j(n) = n \quad \big(\text{or } e^{\mathrm{T}}T(n) = n\big) \tag{4}$$

$$D(n) = MT(n) - Y(n) \tag{5}$$

for any nonnegative integer $n$, where $(\cdot)^{\mathrm{T}}$ denotes the transpose, $e$ is the all-one column vector, and $I_N$ is the $N$-by-$N$ identity matrix. Moreover, if the scheduling is non-idling, then we also have

$$Y_i(n) - Y_i(n-1) = \begin{cases} 1 & \text{if } Z_i(n-1) = 0 \text{ and} \\ & \displaystyle\sum_{j:\, i \in m_j} (T_j(n) - T_j(n-1)) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

for $1 \le i \le N$ and $n = 1, 2, 3, \ldots$. All of the variables take nonnegative integers in each component, and $\{E_i(n)\}$, $\{A_i(n)\}$, $\{D_i(n)\}$, $\{T_j(n)\}$, $\{Y_i(n)\}$ are all nondecreasing in $n$ for any $i$ and $j$. Also we assume the initial conditions are

$$E(0) = A(0) = D(0) = Y(0) = 0 \text{ and } T(0) = 0. \tag{7}$$

For the LQF policy, we have in addition to (2), (3), (4), (5), (6) and (7):

$$T_j(n) - T_j(n-1) = 1 \Rightarrow m_j \in \mathrm{LQF}(Z(n-1)), \tag{8}$$

where $\mathrm{LQF}(Z)$ is the set of possible LQF maximal schedules given queue length vector $Z$. We assume that the schedule is always maximal regardless of the queues being empty or not, so $\mathrm{LQF}(Z) \subseteq \{m_1, m_2, \ldots, m_r\}$.

### 3.4 Fluid Limits

We define the scaled systems based on the queueing network process for each sample path, and show that the scaled systems converge along some subsequence to deterministic systems called fluid limits.

We first extend the definition of $\mathbb{X}$ for arbitrary nonnegative time $t \in \mathbb{R}_+$ by piecewise linear interpolation

$$\mathbb{X}(t) = (1 + \lfloor t \rfloor - t)\mathbb{X}(\lfloor t \rfloor) + (t - \lfloor t \rfloor)\mathbb{X}(\lfloor t \rfloor + 1),$$

where $\lfloor t \rfloor$ is the largest integer less than or equal to $t$. Then $\mathbb{X}$ is an $\bar{\mathcal{X}}$-valued stochastic process with $\bar{\mathcal{X}} = \mathbb{R}_+^{5N+r}$, and is continuous for $t \in \mathbb{R}_+$ given any fixed sample path $\omega \in \Omega$.

Let $|\cdot|$ be the $L^1$-norm of $\mathcal{X}$. Fix $\omega \in \Omega$, and let $\mathbb{X}^x(t)$ be the queueing network process with initial state $\mathbb{X}(0) = x$ for $x \in \mathcal{X}^0 = \{y \in \mathcal{X} \mid |y| > 0\}$ and define the *scaled system*

$$\bar{\mathbb{X}}^x(t) = \frac{1}{|x|}\mathbb{X}^x(|x|t).$$

We then have the following proposition giving the existence of the fluid limits, which is similar to Theorem 4.1 in Dai [6].

**Proposition 1** *For a work-conserving scheduling policy, for almost any sample path $\omega \in \Omega$ and any sequence of initial states $\{x_k, k \in \mathbb{N}\}$ with $x_k \in \mathcal{X}^0$ for all $k$ and $|x_k| \to \infty$ as $k \to \infty$, there exists a subsequence $\{x_{k_p}, p \in \mathbb{N}\}$ with $|x_{k_p}| \to \infty$ as $p \to \infty$ such that*

$$\bar{\mathbb{X}}^{x_{k_p}}(0) \to \bar{\mathbb{X}}(0) \quad \textit{as } p \to \infty$$

*and*

$$\bar{\mathbb{X}}^{x_{k_p}} \to \bar{\mathbb{X}} \quad \textit{u.o.c. as } p \to \infty$$

*for some $\bar{\mathbb{X}}\colon \mathbb{R}_+ \to \bar{\mathcal{X}}$ with $\bar{\mathbb{X}}(t) = (\bar{Z}(t), \bar{E}(t), \bar{A}(t), \bar{D}(t), \bar{T}(t), \bar{Y}(t))$ for any $t \in \mathbb{R}_+$, where "u.o.c." stands for uniform convergence over compact sets [16]. Furthermore, for any $t \in \mathbb{R}_+$,*

$$\bar{A}(0) = \bar{D}(0) = \bar{Y}(0) = 0 \textit{ and } \bar{T}(0) = 0 \tag{9}$$

$$\bar{A}(t) = \alpha t + (R + I_N)\bar{D}(t) \tag{10}$$

$$\bar{Z}(t) = \bar{Z}(0) + \bar{A}(t) - \bar{D}(t) \tag{11}$$

$$e^{\mathrm{T}}\bar{T}(t) = t \tag{12}$$

$$\bar{D}(t) = M\bar{T}(t) - \bar{Y}(t) \tag{13}$$

*and*

$$\int_0^\infty \bar{Z}_i(t)\mathrm{d}\bar{Y}_i = 0 \quad i = 1, 2, \ldots, N. \tag{14}$$

*Moreover, all components of $\bar{\mathbb{X}}$ are absolutely continuous [16] because they are Lipschitz continuous, and $\{\bar{A}_i(t)\}$, $\{\bar{D}_i(t)\}$, $\{\bar{T}_j(t)\}$ and $\{\bar{Y}_i(t)\}$ are nondecreasing in $t$ for all $i$ and $j$.*

*Particularly, the fluid limits under LQF satisfy*

$$\frac{\mathrm{d}\bar{T}_j}{\mathrm{d}t}(t) > 0 \Rightarrow m_j \in \mathrm{LQF}(\bar{Z}(t)) \quad j = 1, 2, \ldots, r, \tag{15}$$

*where $\mathrm{LQF}(\bar{Z})$ is the set of LQF schedules for the vector $\bar{Z}$, and $t$ is assumed regular so the derivatives exist.* ◇

*Remark.* Basically, (9) is the initial condition assumption. (10) says the arrival rates consist of exogenous part and internal part. (11) is the queue evolution equation. (12) comes from the fact that in any period of time $(t, t+\delta)$ of the scaled systems the total increase of the cumulative service time is at most $\delta$. (13) gives the relation among departures, serving time of schedules and idling time. (14) means a link can be idle (when it is chosen by the scheduler) only if the queue length at the link is 0. (15) states that only maximal schedules satisfying the LQF property given the queue fluids $\bar{Z}(t)$ can be chosen at time $t$, but it does not specify the fractions of the schedules that LQF could choose. The proof is standard and thus omitted. Interested readers can refer to Dai and Prabhakar [7], Shah and Wischik [17] and Chen and Yao [4].

### 3.5 Transient States with Dominating Fluids

We first identify a set of transient states of the space of the queue fluid vectors. Let

$$
\begin{aligned}
B_1 &= \{\bar{Z} \in \mathbb{R}_+^N \mid \bar{Z}_1 > \bar{Z}_2\} \\
B_2 &= \{\bar{Z} \in \mathbb{R}_+^N \mid \bar{Z}_2 > \bar{Z}_1, \bar{Z}_2 > \bar{Z}_3\} \\
&\vdots \\
B_{N-1} &= \{\bar{Z} \in \mathbb{R}_+^N \mid \bar{Z}_{N-1} > \bar{Z}_{N-2}, \bar{Z}_{N-1} > \bar{Z}_N\} \\
B_N &= \{\bar{Z} \in \mathbb{R}_+^N \mid \bar{Z}_N > \bar{Z}_{N-1}\}
\end{aligned}
$$

and let

$$
B = \bigcup_{i=1}^{N} B_i.
$$

Then we have

$$
\begin{aligned}
\mathbb{R}_+^N \backslash B = \{\bar{Z} \in \mathbb{R}_+^N \mid \bar{Z}_1 \le \bar{Z}_2, \bar{Z}_{N-1} \ge \bar{Z}_N,& \\
\bar{Z}_i \le \max\{\bar{Z}_{i-1}, \bar{Z}_{i+1}\} \text{ for } 2 \le i \le N-1\}&.
\end{aligned}
$$

So $B$ is the set of queue fluid vectors such that some queue strictly dominates all of its neighbors (one link has at most two neighbors in linear networks), while $\mathbb{R}_+^N \backslash B$ is the set of queue length vectors without any queue strictly dominating all of its neighbors. We then have the following lemma.

**Lemma 1** *$B$ is transient. Formally, given $\alpha_i < 1$ for any $i \in \{1, 2, \dots, N\}$, for any initial conditions $\bar{Z}(t_0) \in \mathbb{R}_+^N$ at time $t_0$, we have $\bar{Z}(t) \notin B$ for any $t \ge t_0 + \frac{\max_i \bar{Z}_i(t_0)}{(1-\max_i \alpha_i)}$ under LQF.* ◇

*Remark.* The outline of the proof is as follows, and the proof can be found in Appendix A.

1. If $\bar{Z} \in B$, then there are no adjacent dominating nodes.
2. Each dominating node loses its domination in time $\frac{\max_i \bar{Z}_i(t_0)}{(1-\max_i \alpha_i)}$.
3. Once a node loses domination, it cannot regain it.

### 3.6 Stability of the First Fluid $\bar{Z}_1$

We now further divide $\mathbb{R}_+^N \backslash B$ into several partitions:

$$C_0 = \{\bar{Z} \in \mathbb{R}_+^N \backslash B \mid \bar{Z}_1 = 0\}$$

$$C_1 = \{\bar{Z} \in \mathbb{R}_+^N \backslash B \mid 0 < \bar{Z}_1 = \bar{Z}_2\}$$

$$C_2 = \{\bar{Z} \in \mathbb{R}_+^N \backslash B \mid 0 < \bar{Z}_1 < \bar{Z}_2 = \bar{Z}_3\}$$

$$\vdots$$

$$C_{N-1} = \{\bar{Z} \in \mathbb{R}_+^N \backslash B \mid 0 < \bar{Z}_1 < \cdots < \bar{Z}_{N-1} = \bar{Z}_N\}.$$

Then $\{C_0, C_1, \ldots, C_{N-1}, B\}$ forms a partition of $\mathbb{R}_+^N$. We then use the following two lemmas to show $C_1$, $C_2$, $\ldots$, $C_{N-1}$ are all transient under admissible arrival rates, so the system has to eventually go to state $C_0$ where $\bar{Z}_1$ stays at 0.

**Lemma 2** *If the arrival rate vector $\alpha$ is admissible, then there exists $\epsilon > 0$ such that for any regular time $t_1 \geq \frac{\max_i \bar{Z}_i(0)}{1-\max_i \alpha_i}$ and $\bar{Z}(t_1) \notin C_0$ we have*

$$\frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) \leq -\epsilon.$$

◇

*Remark.* The idea of the proof is that for any sufficiently large regular time $t_1$ we show that if the fluid of the first queue is positive, then it must decrease with lower-bounded rate. Hence the first fluid reaches zero eventually.

*Proof* Let $t' = \frac{\max_i \bar{Z}_i(0)}{1-\max_i \alpha_i}$. Then by Lemma 1, we have $\bar{Z}(t) \notin B$ for any $t \geq t'$. We let

$$\bar{W}_1(t) = \bar{Z}_1(t)$$

and

$$\bar{W}_i(t) = \bar{Z}_i(t) - \bar{Z}_{i-1}(t) \quad i = 2, 3, \ldots, N.$$

We further let

$$J_0(t) = \{j \mid \bar{W}_j(t) = 0\}$$

and for a regular time $t$,

$$J(t) = \left\{ j \in J_0(t) \;\middle|\; \frac{\mathrm{d}\bar{W}_j}{\mathrm{d}t}(t) = 0 \right\}.$$

Note that $\bar{Z}(t) \in \mathbb{R}_+^N \backslash B$ implies $J_0(t) \neq \emptyset$. We claim that $J(t)$ is also nonempty in the following proposition, the proof of which can be found in Appendix B.

**Proposition 2** *For any $t \geq t'$, we have $J(t) \neq \emptyset$.* ◇

Now we fix a regular time $t_1 \geq t'$ with $\bar{Z}_1(t_1) > 0$ and let

$$u = \min_{j \in J(t_1)} j.$$

Then $u \geq 2$. Let the service rate on link $l_i$ at time $t$ be $\mu_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_i(t)$ for regular time $t$ and any $i \in \{1, 2, \ldots, N\}$. Then we claim that the service rates up to $u$ at time $t_1$ satisfy the following proposition, the proof of which is presented in Appendix C.

**Proposition 3** *For $i = 1, 2, \ldots, u-2$, $\mu_i(t_1) = \mu_{i+2}(t_1)$.* ◇

Due to one-hop interference model, we have

$$\mu_1(t_1) + \mu_2(t_1) = 1$$

since at each time slot in the real system either link $l_1$ or link $l_2$ must be scheduled. Then by the definition of $u$, we have $\frac{\mathrm{d}\bar{Z}_{u-1}}{\mathrm{d}t}(t_1) = \frac{\mathrm{d}\bar{Z}_u}{\mathrm{d}t}(t_1)$, i.e.,

$$\mu_{u-2}(t_1) + \alpha_{u-1} - \mu_{u-1}(t_1) = \mu_{u-1}(t_1) + \alpha_u - \mu_u(t_1)$$

where $\mu_0(t_1) = 0$ by convention. Then if $u = 2$, we have

$$\begin{aligned} &\begin{cases} \mu_1(t_1) + \mu_2(t_1) = 1 \\ \alpha_1 - \mu_1(t_1) = \mu_1(t_1) + \alpha_2 - \mu_2(t_1) \end{cases} \\ \Rightarrow &\begin{cases} \mu_1(t_1) = \frac{1}{3} + \frac{1}{3}\alpha_1 - \frac{1}{3}\alpha_2 \\ \mu_2(t_1) = \frac{1}{3} - \frac{1}{3}\alpha_1 + \frac{1}{3}\alpha_2 \end{cases} \\ \Rightarrow &\frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) = \alpha_1 - \mu_1(t_1) = -\frac{1}{3} + \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2. \end{aligned}$$

Similarly, if $u = 3$,

$$\begin{aligned} &\begin{cases} \mu_1(t_1) + \mu_2(t_1) = 1 \\ \mu_1(t_1) = \mu_3(t_1) \\ \mu_1(t_1) + \alpha_2 - \mu_2(t_1) = \mu_2(t_1) + \alpha_3 - \mu_3(t_1) \end{cases} \\ \Rightarrow &\begin{cases} \mu_1(t_1) = \mu_3(t_1) = \frac{1}{2} - \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 \\ \mu_2(t_1) = \frac{1}{2} + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3 \end{cases} \\ \Rightarrow &\frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) = \alpha_1 - \mu_1(t_1) = -\frac{1}{2} + \alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3, \end{aligned}$$

and if $u = 4$ we have,

$$\begin{cases} \mu_1(t_1) + \mu_2(t_1) = 1 \\ \mu_1(t_1) = \mu_3(t_1) \\ \mu_2(t_1) = \mu_4(t_1) \\ \mu_2(t_1) + \alpha_3 - \mu_3(t_1) = \mu_3(t_1) + \alpha_4 - \mu_4(t_1) \end{cases}$$
$$\Rightarrow \begin{cases} \mu_1(t_1) = \mu_3(t_1) = \frac{1}{2} + \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4 \\ \mu_2(t_1) = \mu_4(t_1) = \frac{1}{2} - \frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_4 \end{cases}$$
$$\Rightarrow \frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) = \alpha_1 - \mu_1(t_1) = -\frac{1}{2} + \alpha_1 - \frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_4.$$

We can then get the derivative of $\bar{Z}_1(\cdot)$ at $t_1$ as

$$\frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) = \begin{cases} -\frac{1}{3} + \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 & \text{if } u = 2 \\ -\frac{1}{2} + \alpha_1 + \frac{1}{4}\alpha_{u-1} - \frac{1}{4}\alpha_u & \text{if } u = 3, 5, \ldots \\ -\frac{1}{2} + \alpha_1 - \frac{1}{4}\alpha_{u-1} + \frac{1}{4}\alpha_u & \text{if } u = 4, 6, \ldots \end{cases}.$$

Since $\alpha$ is admissible, we have [19]

$$-R^{-1}\alpha < M\gamma \tag{16}$$

for some convex combination coefficients $\gamma$. Then by (1), we have

$$-R^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Note that the $i$th row of $-R^{-1}\alpha$ is the *total workload* of link $l_i$, while the $i$th row of $M\gamma$ is the service rate on link $l_i$. Since the total workload of the last two links are $\sum_{i=1}^{N-1}\alpha_i$ and $\sum_{i=1}^{N}\alpha_i$ respectively, and exactly one of the last two links must be chosen at each time slot, we have by combining the last two rows of (16)

$$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + \alpha_N < 1.$$

So $\frac{\mathrm{d}\bar{Z}_1}{\mathrm{d}t}(t_1) \leq -\epsilon$, where

$$\begin{aligned}\epsilon = \min \Big\{ & \frac{1}{3} - \frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2, \\ & \frac{1}{2} - \alpha_1 - \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3, \\ & \frac{1}{2} - \alpha_1 + \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4, \\ & \dots, \\ & \frac{1}{2} - \alpha_1 - \frac{1}{4}(-1)^{N-1}\alpha_{N-1} - \frac{1}{4}(-1)^N\alpha_N \Big\} \\ > 0. & \end{aligned}$$

□

**Corollary 1** *Given the initial conditions and arrival rates in Lemma 2, there exists* $\epsilon > 0$ *such that* $\bar{Z}_1(t) = 0$ *for any* $t \geq \frac{\bar{Z}_1(0)}{\epsilon} + \frac{\max_i \bar{Z}_i(0)}{1-\max_i \alpha_i}\left(\frac{1}{\epsilon} + 1\right)$. ◇

*Remark.* This comes directly from Lemma 2 and a similar proof of Proposition 5 in Appendix A. Basically, $\bar{Z}_1(t)$ has to drop to 0, after which it cannot rise since otherwise the negative derivative forces it to go back to 0.

## 3.7 Coupled-Network Argument

Based on Corollary 1, we use induction and a coupled-network argument to show the following lemma stating the stability of the fluid system, which leads to the stability of the original queueing system using a similar argument as presented by Dai [6].

**Lemma 3** *Given the initial conditions and arrival rates in Lemma 2, there exists* $c_3 > 0$ *such that* $\bar{Z}_i(t) = 0$ *for any* $t \geq \max_j \bar{Z}_j(0)c_3$ *and any* $i = 1, 2, \dots, N$. ◇

*Proof* We use induction. First, by Corollary 1, there exists $\tilde{c} > 0$ such that $\bar{Z}_1(t) = 0$ for any $t \geq \max_j \bar{Z}_j(0)\tilde{c}$. Now suppose there exists $c$ and $k$ such that $\bar{Z}_i(t) = 0$ and $\bar{Z}_{k+1}(t) > 0$ for any $t \geq \max_j \bar{Z}_j(0)c$ and $i \leq k$. We consider a coupled linear network under the LQF scheduling with $N - k$ links, initial fluids $\bar{Z}'_i(\max_j \bar{Z}_j(0)c) = \bar{Z}_{i+k}(\max_j \bar{Z}_j(0)c)$ for $1 \leq i \leq N - k$, and arrival rates

$$\alpha'_1 = \alpha_1 + \alpha_2 + \dots + \alpha_{k+1}$$

and

$$\alpha'_j = \alpha_{k+j} \qquad j = 2, 3, \dots, N - k.$$

Thus $\{\bar{Z}'_i(t), 1 \leq i \leq N\}$ are the fluids of the original network with the first $k + 1$ links combined into one link. Since the fluids satisfy $\bar{Z}_{k+1}(t) > \bar{Z}_k(t)$, we have that the queue length at link $l_{k+1}$ is larger than that at link $l_k$ in the actual system. Then by the LQF scheduling, the schedule of the first $k$ links does not affect the schedule of the last $N-k$ links. Also notice that the fluid arrival to $\bar{Z}_{k+1}(t)$ is $\alpha_1+\alpha_2+\cdots+\alpha_{k+1}=\alpha'_1$ since all fluids $\bar{Z}_i(t)$'s prior to $\bar{Z}_{k+1}(t)$ remain zero, transferring their exogenous arrival to $\bar{Z}_{k+1}(t)$. Hence, $\bar{Z}_{i+k}(t)=\bar{Z}'_i(t)$ for all $t\geq \max_j \bar{Z}_j(0)c$.

Taking the last $N-k$ rows of (16), we can get

$$-R'^{-1}\alpha' < M'\gamma',$$

where $R'$ is the routing matrix for the coupled network, $M'$ is the maximal scheduling matrix of the coupled network, and $\gamma'$ is a set of convex combination coefficients induced from $\gamma$. Note that $M'$ consists of the maximal columns of the matrix formed by the last $N-k$ rows of $M$. Hence $\alpha'$ is also admissible. Let $\bar{Z}'_{\max}=\max_i \bar{Z}'_i(0)$. Then by the Lipschitz continuity we have $\bar{Z}'_{\max}\leq \max_j \bar{Z}_j(0)c_4$ for some $c_4>0$. Again by Corollary 1, there exists $c_5>0$ such that $\bar{Z}'_1(t)=0$ for any $t\geq \bar{Z}'_{\max}c_5$. Consequently, $\bar{Z}_{k+1}(t)=\bar{Z}'_1(t)=0$ is also true for $t\geq \max_j \bar{Z}_j(0)c_4c_5$. By mathematical induction, we get that there exists $c_3>0$ such that $\bar{Z}_i(t)=0$ for any $t\geq \max_j \bar{Z}_j(0)c_3$ and $1\leq i\leq N$. □

### 3.8 Uniform Integrability

According to Lemma 4.5 in Dai [6], we now have for sufficiently large $t$,

$$\frac{1}{n}\max_i Z_i(nt)\to 0$$

as $n\to\infty$. Following a similar argument given in Dimakis and Walrand [8], to get the stability of the system, we only need to show

$$\mathbb{E}\left(\frac{1}{n}\max_i Z_i(nt)\right)\to 0$$

as $n\to\infty$. I.e., we only need to show $\left\{\frac{1}{n}\max_i Z_i(nt), n\in\mathbb{N}\right\}$ are uniformly integrable (UI). Note

$$\begin{aligned}\frac{1}{n}\max_i Z_i(nt) &\leq \frac{1}{n}\sum_i Z_i(nt)\\ &\leq \frac{1}{n}\sum_i E_i(nt).\end{aligned}$$

Then by the law of large numbers, $\frac{1}{n}\sum_i E_i(nt)$ converges to $\sum_i \alpha_i t$ in probability as $n\to\infty$. Also note

$$\mathbb{E}\left(\frac{1}{n}\sum_i E_i(nt)\right)=\sum_i \alpha_i t.$$

Then by Theorem 4.5.4 in Chung [5] we have UI of $\{\frac{1}{n}\sum_i E_i(nt)\}$ and thus that of $\{\frac{1}{n}\max_i Z_i(nt)\}$.

## 4 Conclusions

We studied the stability of the longest-queue-first scheduling policy in wireless networks with multihop traffic flows and the one-hop interference model. Using fluid techniques, we proved that LQF is throughput optimal in this scenario. The proof itself is an interesting contribution and can be useful when considering similar fluid systems since we focused on state transition instead of an explicit Lyapunov function. The result may also be a first step to understand the stability performance of LQF in general networks with multihop traffic flows.

## ACKNOWLEDGMENTS

We thank Sanjay Shakkottai, R. Srikant and Alexander L. Stolyar for many useful comments.

## A Proof of Lemma 1

We first notice that if $\bar{Z} \in B$, then there are no adjacent dominating nodes; i.e., if $\bar{Z} \in B_i$ for some $i \in \{1, 2, \ldots, N\}$, then $\bar{Z} \notin B_{i-1}$ and $\bar{Z} \notin B_{i+1}$. Let the dominating set at time $t$ be

$$I_{\mathrm{dom}}(t) = \{i \in \{1, 2, \ldots, N\} \mid \bar{Z}(t) \in B_i\}.$$

Then we can easily check that $I_{\mathrm{dom}}(t) \subseteq \bigcap \mathrm{LQF}(\bar{Z}(t))$; i.e., all dominating links must be scheduled by LQF at time $t$. Due to the one-hop interference, there is no internal arrival to a scheduled link. Then for regular time $t$ and any $i \in I_{\mathrm{dom}}(t)$,

$$\frac{\mathrm{d}A_i}{\mathrm{d}t}(t) = \frac{\mathrm{d}E_i}{\mathrm{d}t}(t) = \alpha_i$$

and

$$\frac{\mathrm{d}D_i}{\mathrm{d}t}(t) = 1,$$

while

$$\frac{\mathrm{d}D_{i-1}}{\mathrm{d}t}(t) = \frac{\mathrm{d}D_{i+1}}{\mathrm{d}t}(t) = 0.$$

Thus,

$$\frac{\mathrm{d}\bar{Z}_i}{\mathrm{d}t}(t) - \frac{\mathrm{d}\bar{Z}_{i-1}}{\mathrm{d}t}(t) \leq \alpha_i - 1 \tag{17}$$

and

$$\frac{\mathrm{d}\bar{Z}_i}{\mathrm{d}t}(t) - \frac{\mathrm{d}\bar{Z}_{i+1}}{\mathrm{d}t}(t) \leq \alpha_i - 1. \tag{18}$$

Let $\bar{Z}_{\max} = \max_j \bar{Z}_j(0)$. We now present two propositions to complete the proof of Lemma 1.

**Proposition 4** *There exists $t_1 \in (t_0, t_0 + \bar{Z}_{\max}/(1-\alpha_i)]$ such that $\bar{Z}(t_1) \notin B_i$; i.e., link $l_i$ is not dominating anymore at some time before $t_0 + \bar{Z}_{\max}/(1-\alpha_i)$.* ◇

*Proof* Indeed, if link $l_i$ remains dominating up to (and including) time $t_0 + \bar{Z}_{\max}/(1-\alpha_i)$, then by (17), (18) and the absolute continuity of the fluids, we would have for any adjacent link $l_j$ of $l_i$,

$$\left[\bar{Z}_i\left(t_0 + \frac{\bar{Z}_{\max}}{1-\alpha_i}\right) - \bar{Z}_j\left(t_0 + \frac{\bar{Z}_{\max}}{1-\alpha_i}\right)\right] - \left[\bar{Z}_i(t_0) - \bar{Z}_j(t_0)\right] \leq -\bar{Z}_{\max}.$$

Hence

$$\bar{Z}_i\left(t_0+\frac{\bar{Z}_{\max}}{1-\alpha_i}\right)-\bar{Z}_j\left(t_0+\frac{\bar{Z}_{\max}}{1-\alpha_i}\right)\leq \bar{Z}_i(t_0)-\bar{Z}_{\max}\leq 0.$$

Then by continuity, there is some $t_1\in(t_0,t_0+\bar{Z}_{\max}/(1-\alpha_i)]$ such that $\bar{Z}_i(t_1)-\bar{Z}_j(t_1)=0$, which contradicts our assumption that link $l_i$ remains dominating up to $t_0+\bar{Z}_{\max}/(1-\alpha_i)$. This completes the proof of the proposition. □

Therefore, for any $i\in I_{\mathrm{dom}}(t_0)$, there exists $t_1\in(t_0,t_0+\bar{Z}_{\max}/(1-\max_j\alpha_j)]$ such that $\bar{Z}(t_1)\notin B_i$.

**Proposition 5** *If $\bar{Z}(t_1)\notin B_i$, then $\bar{Z}(t)\notin B_i$ for any $t\geq t_1$.* ◇

*Proof* Indeed, if $\bar{Z}(t_2)\in B_i$ for some $t_2>t_1$, let $t_3=\sup\{t<t_2\mid \bar{Z}(t)\notin B_i\}$. Then by Lipschitz continuity $\bar{Z}_i(t_3)=\bar{Z}_j(t_3)$ for some neighbor $l_j$ of link $l_i$ and $t_3<t_2$. Since $\frac{\mathrm{d}}{\mathrm{d}t}(\bar{Z}_i(t)-\bar{Z}_j(t))\leq\max_k\alpha_k-1$ for almost all $t\in[t_3,t_2]$, we have

$$\bar{Z}_i(t_2)-\bar{Z}_j(t_2)\leq\bar{Z}_i(t_3)-\bar{Z}_j(t_3)+(t_2-t_3)(\max_k\alpha_k-1)\leq 0,$$

which contradicts the assumption that $\bar{Z}(t_2)\in B_i$. Hence, $\bar{Z}(t)\notin B_i$ for any $t\geq t_1$. □

Considering all $i$, we have $\bar{Z}(t)\notin B$ for any $t\geq t_0+\bar{Z}_{\max}/(1-\max_j\alpha_j)$.

## B Proof of Proposition 2

To show this proposition by contradiction, we suppose $J(t)=\emptyset$.

We first notice that for $j_1=\min\{j\mid j\in J_0(t)\}$ we must have $\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_{j_1}(t)>0$. If this is not the case, we would have $\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_{j_1}(t)<0$ and $j_1\geq 2$. Then it would follow that there exists some $\delta>0$ such that for any $s\in(t,t+\delta)$ we have $\bar{Z}_1(s)>\bar{Z}_2(s)$ if $j_1=2$, and $\bar{Z}_{j_1-1}(s)>\max\{\bar{Z}_{j_1-2}(s),\bar{Z}_{j_1}(s)\}$ if $j_1>2$, which implies $\bar{Z}(s)\in B$, a contradiction.

We then conclude that if all $j\in\{1,2,\ldots,k\}\cap J_0(t)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_j(t)>0,$$

then either $k+1\notin J_0(t)$ or

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_{k+1}(t)>0.$$

If this is not the case, there would exist $\delta>0$ such that for any $s\in(t,t+\delta)$, we have $\bar{Z}_k(s)>\max\{\bar{Z}_{k-1}(s),\bar{Z}_{k+1}(s)\}$ if $\bar{W}_k(t)\geq 0$, or $\bar{Z}_j(s)>\max\{\bar{Z}_{j-1}(s),\bar{Z}_{j+1}(s)\}$ for some $j<k$ otherwise, either of which leads to a contradiction.

By induction we have $\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_j(t)>0$ for all $j\in J_0(t)$, which also leads to contradiction since by letting $j_2=\max\{j\mid j\in J_0(t)\}$ there exists $\delta>0$ such that for any $s\in(t,t+\delta)$ we have $\bar{Z}_N(s)>\bar{Z}_{N-1}(s)$ if $j_2=N$, and $\bar{Z}_{j_2}(s)>\max\{\bar{Z}_{j_2-1}(s),\bar{Z}_{j_2+1}(s)\}$ if $j_2\neq N$. Then $\bar{Z}(s)\in B$, which is a contradiction. This completes the proof of Proposition 2.

## C Proof of Proposition 3

Note that $u\geq 2$. By definition of $u$, $\bar{W}_u(t_1)=\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_u(t_1)=0$, i.e., $\bar{Z}_u(t_1)=\bar{Z}_{u-1}(t_1)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\bar{Z}_u(t_1)=\frac{\mathrm{d}}{\mathrm{d}t}\bar{Z}_{u-1}(t_1)$.

We first claim that there exists $\delta>0$ such that for any $t\in(t_1,t_1+\delta)$,

$$0<\bar{Z}_1(t)<\bar{Z}_2(t)<\cdots<\bar{Z}_{u-1}(t).$$

Indeed, if $J_0(t_1) \cap \{1, 2, \ldots, u-1\} = \emptyset$, then $\bar{W}_i(t_1) \neq 0$ for $i = 1, 2, \ldots, u-1$. Otherwise for any $j \in J_0(t_1) \cap \{1, 2, \ldots, u-1\}$, by the definitions of $u$ and $J_0(\cdot)$ we have $\bar{W}_j(t_1) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_j(t_1) \neq 0$, so there exists $\delta_j > 0$ such that $\bar{W}_j(t) \neq 0$ for any $t \in (t_1, t_1 + \delta_j)$. In either case, there exists $\delta > 0$ such that $\bar{W}_i(t) \neq 0$ for any $t \in (t_1, t_1 + \delta)$ and any $i \in \{1, 2, \ldots, u-1\}$. If $\bar{W}_i(t) < 0$ for some $i \in \{1, 2, \ldots, u-1\}$ and some $t \in (t_1, t_1 + \delta)$, then $\bar{Z}_j(t) > \max\{\bar{Z}_{j-1}(t), \bar{Z}_{j+1}(t)\}$ for some $j \leq i$, which contradicts Lemma 1. Hence $\bar{W}_i(t) > 0$ for $i = 1, 2, \ldots, u-1$; i.e., $0 < \bar{Z}_1(t) < \bar{Z}_2(t) < \cdots < \bar{Z}_{u-1}(t)$. The claim then follows.

In the actual system with this strict order of queues, either all odd links up to $l_u$ get scheduled at a time slot, or all even links up to $l_u$ get scheduled. Then in our fluid limits we would have

$$\mu_i(t) = \mu_{i+2}(t) \quad \forall i \in \{1, 2, \ldots, u-2\}$$

for any regular time $t \in (t_1, t_1 + \delta)$. Then by the absolute continuity,

$$\bar{D}_3(t) - \bar{D}_1(t) = \bar{D}_3(t_1) - \bar{D}_1(t_1) + \int_{t_1}^{t} (\mu_3(s) - \mu_1(s))\mathrm{d}s = \bar{D}_3(t_1) - \bar{D}_1(t_1).$$

By the definition of derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{D}_3(t_1) - \bar{D}_1(t_1)) = \lim_{t \to t_1^+} \frac{(\bar{D}_3(t) - \bar{D}_1(t)) - (\bar{D}_3(t_1) - \bar{D}_1(t_1))}{t - t_1} = 0.$$

So $\mu_1(t_1) = \mu_3(t_1)$. Similarly, we have $\mu_i(t_1) = \mu_{i+2}(t_1)$ for $i = 1, 2, \ldots, u-2$.

## References


1. Birand, B., Chudnovsky, M., Ries, B., Seymour, P., Zussman, G., Zwols, Y.: Analyzing the performance of greedy maximal scheduling via local pooling and graph theory. IEEE/ACM Trans. Netw. **20**, 163–176 (2012)
2. Brzezinski, A., Zussman, G., Modiano, E.: Local pooling conditions for joint routing and scheduling. In: Proc. Inf. Theory and Appl. Workshop (ITA), pp. 499–506. San Diego, CA (2008)
3. Bui, L.X., Srikant, R., Stolyar, A.: A novel architecture for reduction of delay and queueing structure complexity in the back-pressure algorithm. IEEE/ACM Trans. Netw. **19**, 1597–1609 (2011)
4. Chen, H., Yao, D.D.: Fundamentals of Queueing Networks: Performance, Asymptotics, and Optimization, *Stochastic Modelling and Applied Probability*, vol. 46. Springer, New York, NY (2001)
5. Chung, K.L.: A Course in Probability Theory, third edn. Academic Press, San Diego, CA (2001)
6. Dai, J.G.: On positive Harris recurrence of multiclass queueing networks: A unified approach via fluid limit models. Ann. Appl. Probab. **5**(1), 49–77 (1995)
7. Dai, J.G., Prabhakar, B.: The throughput of data switches with and without speedup. In: Proc. IEEE Int. Conf. Comput. Commun. (INFOCOM), vol. 2, pp. 556–564. Tel Aviv, Israel (2000)
8. Dimakis, A., Walrand, J.: Sufficient conditions for stability of longest queue first scheduling: Second-order properties using fluid limits. Adv. in Appl. Probab. **38**(2), 505–521 (2006)
9. Hellings, T., Borst, S.C., van Leeuwaarden, J.S.: Tandem queueing networks with neighbor blocking and back-offs. Queueing Syst. **68**(3–4), 321–331 (2011)
10. Joo, C., Lin, X., Shroff, N.B.: Greedy maximal matching: Performance limits for arbitrary network graphs under the node-exclusive interference model. IEEE Trans. Autom. Control **54**, 2734–2744 (2009)
11. Joo, C., Lin, X., Shroff, N.B.: Understanding the capacity region of the greedy maximal scheduling algorithm in multihop wireless networks. IEEE/ACM Trans. Netw. **17**, 1132–1145 (2009)

12. Kang, X., Jaramillo, J.J., Ying, L.: Stability of longest-queue-first scheduling in linear wireless networks with multihop traffic and one-hop interference. In: Proc. IEEE Conf. Decision and Control (CDC), pp. 3312–3317. Florence, Italy (2013)
13. Leconte, M., Ni, J., Srikant, R.: Improved bounds on the throughput efficiency of greedy maximal scheduling in wireless networks. IEEE/ACM Trans. Netw. **19**, 709–720 (2011)
14. Maguluri, S., Hajek, B., Srikant, R.: The stability of longest-queue-first scheduling with variable packet sizes. IEEE Trans. Autom. Control **59**, 2295–2300 (2014)
15. Ni, J., Tan, B., Srikant, R.: Q-CSMA: Queue-length-based CSMA/CA algorithms for achieving maximum throughput and low delay in wireless networks. IEEE/ACM Trans. Netw. **20**, 825–836 (2012)
16. Royden, H.L., Fitzpatrick, P.M.: Real Analysis, fourth edn. Prentice Hall (2010)
17. Shah, D., Wischik, D.: Switched networks with maximum weight policies: Fluid approximation and multiplicative state space collapse. Ann. Appl. Probab. **22**(1), 70–127 (2012)
18. Stolyar, A.L.: Large number of queues in tandem: Scaling properties under back-pressure algorithm. Queueing Syst. **67**(2), 111–126 (2011)
19. Tassiulas, L., Ephremides, A.: Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. IEEE Trans. Autom. Control **37**, 1936–1948 (1992)
20. Tassiulas, L., Ephremides, A.: Dynamic scheduling for minimum delay in tandem and parallel constrained queueing models. Ann. Oper. Res. **48**(4), 333–355 (1994)