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Short communication

Short-term fairness and long-term QoS in the Internet

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ABSTRACT

We consider connection-level models of resource allocation in a simple symmetric three-link network, where files arrive into the network according to a Poisson process and the size of each file is exponentially distributed. For the simple network under consideration, we derive an optimal resource allocation policy which minimizes the steady-state expected number of files in the network. Using this result, we show that, in a heavy traffic regime, the mean file-transfer delay under the proportionally fair policy is at most 1.5 times the delay under the optimal policy. Simulation results indicate that the gap is even smaller.

1. Introduction

Resource allocation algorithms for the Internet are designed to ensure fair resource allocation among the large amount of users who are present in the network. Such algorithms are often designed under the assumption of fixed number of users, which we refer to as a *static model*. The study of distributed algorithms for the static model was initiated in [1] and has been studied by a number of researchers (see [2] and the references therein).

In reality, however, the number of users in the network is time-varying: new users arrive into the network, bringing in a certain amount of workload (in this paper, we regard file transfers as the only form of workload), while current users leave the network when their work is completed. The stability of this "more realistic" network, in which the arrivals and departures of files (i.e., of users) are taken into account, has been studied in a number of papers, such as [3–6]. Such models are often called *connection-level models*. In this paper, our goal is to go one step beyond stability and study the impact of resource allocation on the connection-level delay performance of the network.

Specifically, we consider the proportionally fair resource allocation policy introduced in [7]. Even though the proportionally fair resource allocation policy was originally designed for short-term fairness (i.e., for time scales where the number of users in the network is fixed), it has been shown that it can also support the maximum possible connection-level throughput [4,5]. However, the connection-level delay, which is a measure of long-term QoS (Quality of Service), may not be minimized under the proportionally fair policy.

In this paper, to analyze delay performance, we derive an upper bound on the steady-state expected number of files in a symmetric three-link star network under the proportionally fair policy. Further, we compare the delay performances of the proportionally fair policy and an optimal policy that minimizes the steady-state expected number of files in this network. According to Little's law [8], the steady-state expected number of files is proportional to the expected file-transfer delay; hence the policy can also be viewed as being "delay optimal".

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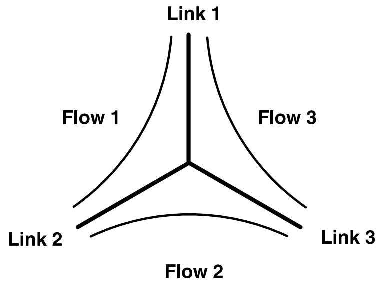

Fig. 1. A symmetric three-link star network.

The results for the optimal resource allocation algorithm in this paper complement the results in [9], in which the authors considered a similar problem for line networks. For our symmetric three-link star network, in addition to deriving the optimal policy and comparing it with the proportionally fair policy using simulations as the authors did for their linear network in [9], we have also derived a lower bound on the steady-state expected number of files under the optimal policy. The lower bound allows us to show that, in a heavy traffic regime, the steady-state expected number of files under the proportionally fair policy is at most 1.5 times as much as the one under the optimal policy.

The result for the upper bound in this paper complements the work of Kang et al. [10] who conjectured the steady-state distribution of the number of files in the network in a heavy traffic regime using a Brownian motion approximation. Our upper bound matches their expression for the symmetric three-link star network. The result in [10] is a generalization of the earlier results on product-form solutions obtained by [3,11].

2. Model and main results

In this paper, we consider a symmetric star network with three links and three types of flows or files as shown in Fig. 1. We assume that files of each type arrive according to independent Poisson processes with rate λ files/s, the file sizes are independent and exponentially distributed, with mean file size $1/\mu$ bits, and the capacity of each link is 1 bit/s. Let $n_i(t)$ denote the number of files of type i in the network at time t, and $x_i(t)$ denote the rate allocated to serve files of type i at time t. Additionally, we assume that the files of the same type equally share the allocated rate, i.e., at time t, a file of type i will be served at rate $x_i(t)/n_i(t)$ if $n_i(t) > 0$. Note that each file's route consists of two links, so a feasible rate allocation $\mathbf{x} = (x_1, x_2, x_3)$ has to satisfy $x_i \geq 0$ and $x_i + x_j \leq 1$ for $1 \leq i, j \leq 3$, $i \neq j$. The feasible set of $\mathbf{x}$ will be denoted as $\mathcal{F}$. We note that star networks are a reasonable approximation of the Internet where most of the congestion is at the edges and the core is relatively uncongested.

The evolution of $\mathbf{n}(t) = (n_1(t), n_2(t), n_3(t))$ can be described by a continuous-time Markov chain. Furthermore, using uniformization [12], we can construct a discrete-time Markov chain $\mathbf{n}[t]$ by sampling the network at certain random times as we will describe now [2,13]. Let ν denote the maximum rate at which events occur in this network, where $\nu = \max_{x\in\mathcal{F}}\{3\lambda + \sum_{i=1}^{3} x_i\mu\}$. It can be easily shown that this maximum rate $\nu = 3\lambda + 3\mu/2$. We sample the system according to a Poisson process of this rate. Specifically, after each sample which we call an event, the next event is an arrival of a file of type i with probability λ/ν, a departure of a file of type i with probability $x_i\mu/\nu$, or the system state is left unchanged with the remaining probability. These events are called arrivals, departures and fictitious departures, respectively. From now on, without loss of generality we assume $\nu = 1$.

For the discrete-time system obtained through uniformization, let $n_i[t]$ denote the number of files of type i during the time from the t-th event to $(t + 1)$-th event, named as *time slot* t. We further let $x_i[t]$ denote the rate allocated to serve flow i at time slot t. It is obvious that the network dynamics at the connection level depends on the resource allocation algorithm that decides $\mathbf{x}[t]$ at each time slot t. One particular resource allocation scheme that has been widely studied is the proportionally fair resource allocation algorithm (referred to as **Algorithm PF** in the rest of the paper), which is defined as follows: At time slot t, files of type i are served at rate $x_i[t]$, where $\mathbf{x}[t] = (x_1[t], x_2[t], x_3[t])$ is the solution of the following optimization problem: $\mathbf{x}[t] = \arg\max_{\mathbf{x}\in\mathcal{F}} \sum_i n_i[t] \log x_i$.

For the star network considered in this paper, we have the following result:

Lemma 1. *The star network is stochastically stable under the proportionally fair algorithm if* $\rho \triangleq \frac{2\lambda}{\mu} < 1$. ◇

The above result was established using a fluid model in [5]. For the purposes of this paper, the approach in [2,13] is useful which, in addition to stability, can also provide performance bound as shown later.

In this paper, we study the steady-state expected number of files in the network, defined as $E\left[\sum_{i=1}^{3} n_i[\infty]\right]$. Note that by Little's law, the file-transfer delay is given by $E\left[\sum_{i=1}^{3} n_i[\infty]\right]/3\lambda$. In Section 3, we analyze the proportionally fair algorithm and obtain the following upper bound on the expected number of files in the network.

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Theorem 2. *Considering the star network with $\rho < 1$, under the proportionally fair algorithm, we have*

$$E\left[\sum_{i=1}^{3} n_i^{PF}[\infty]\right] \leq \frac{3\rho}{1-\rho},$$

where the superscript PF indicates the proportionally fair algorithm. $\diamond$

Note that for the symmetric three-link star network, our upper bound matches the approximation in [10]. Our goal to study the symmetric three-link star network is to further understand how well the proportionally fair resource allocation algorithm performs, compared with an optimal resource allocation algorithm that minimizes the steady-state expected number of files.

To describe our algorithms, we say that file type i is nonempty in time slot t if $n_i[t] > 0$, and empty if $n_i[t] = 0$. In Section 4, we prove that the following algorithm (referred to as **Algorithm OPT** in the rest of the paper) minimizes the steady-state expected number of files for the symmetric three-link star network: Within time slot t, $\mathbf{x}[t]$ is chosen as follows:

(i) If all the three file types are nonempty, then $x_1[t] = x_2[t] = x_3[t] = 0.5$.
(ii) If only two file types are nonempty, then the resources are allocated as follows:
  (a) If one nonempty file type has more files than the other, serve the one with more files at rate one.
  (b) If the two nonempty file types have the same number of files, randomly pick one type and serve it with rate one.
(iii) If only one file type is nonempty, the file type is served at rate one.

Algorithm OPT will be analyzed in Section 4. We will first prove its optimality, and then obtain the following lower bound on the steady-state expected number of files:

Theorem 3. *Considering the symmetric three-link star network with $\rho < 1$, under the optimal algorithm, we have*

$$E\left[\sum_{i=1}^{3} n_i^{OPT}[\infty]\right] \geq \frac{5\rho-1}{2(1-\rho)}. \quad \diamond$$

Remark. From Theorems 2 and 3, we observe that

$$\lim_{\rho\to 1} \frac{E\left[\sum_{i=1}^{3} n_i^{PF}[\infty]\right]}{E\left[\sum_{i=1}^{3} n_i^{OPT}[\infty]\right]} \leq \lim_{\rho\to 1} \frac{6\rho}{5\rho-1} = 1.5.$$

Thus, we can conclude that in the symmetric three-link star network, under the proportionally fair algorithm, the steady-state expected number of files is at most 1.5 times the number under the optimal policy in a heavy traffic regime. Our simulation results in Section 4.3 will confirm the tightness of the upper bound (Theorem 2) in the heavy traffic regime. Simulation results show that the lower bound (Theorem 3) is not tight, which indicates that the delay performance of the proportionally fair algorithm is even better than the theoretical prediction.

3. Proportional fairness

In this section, we prove Theorem 2, i.e., the upper bound on the steady-state expected number of files in the network under the proportionally fair algorithm. An earlier version of this upper bound appeared in a short version of this paper in [14]. The upper bound uses a stochastic Lyapunov drift condition as in [2,13] which in turns uses ideas from [5].

3.1. Upper bound on steady-state expected number of files

Proof of Theorem 2. Since we only consider the proportionally fair algorithm in this proof, we can ignore the superscript without causing any confusion. Define the Lyapunov function $W[t] = \sum_{i=1}^{3} \frac{1}{\lambda} n_i^2[t]$. First of all, we note that

$$E\left[W[t+1] - W[t] | \mathbf{n}[t]\right] = \sum_{i=1}^{3}((n_i[t]+1)^2 - n_i[t]^2) + \sum_{i=1}^{3} \frac{\mu}{\lambda} x_i[t]((n_i[t]-1)^2 - n_i[t]^2)\mathcal{I}_{n_i[t]>0},$$

which implies that

$$\begin{aligned} E\left[W[t+1] - W[t] | \mathbf{n}[0]\right] &= E\left[E\left[W[t+1] - W[t] | \mathbf{n}[t]\right] | \mathbf{n}[0]\right] \\ &= 3 + \sum_{i=1}^{3} \frac{1}{\lambda} \left(E[2(\lambda - \mu x_i[t]) n_i[t] | \mathbf{n}[0]] + E\left[\mu x_i[t] \mathcal{I}_{n_i[t]>0} | \mathbf{n}[0]\right]\right). \end{aligned} \quad (1)$$

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On the other hand, recall that the resource allocation under the proportional fairness is a solution to the following optimization problem: $\max_{\mathbf{x}\in\mathcal{F}} f(\mathbf{x})$, where $f(\mathbf{x}) = \sum_{i=1}^{3} n_i[t]\log x_i$. Since $f(\mathbf{x})$ is concave and $\mathcal{F}$ is a convex set, we have $\nabla f\cdot(\mathbf{y}-\mathbf{x}[t]) \le 0$ for any $\mathbf{y}\in\mathcal{F}$, where $\mathbf{x}[t]$ is the optimal solution given $\mathbf{n}[t]$, which implies that

$$\sum_{i=1}^{3} \frac{n_i[t]}{y_i}(y_i - x_i[t]) \le 0. \tag{2}$$

Since $(0.5, 0.5, 0.5) \in \mathcal{F}$, from inequality (2), we get $\sum_{i=1}^{3} n_i[t](1-2x_i[t]) \le 0$. Next, choosing $\epsilon > 0$ such that $(1+\epsilon)\rho = 2(1+\epsilon)\lambda/\mu = 1$, we get

$$\sum_{i=1}^{3} n_i[t]\left(1 - \frac{\mu}{\lambda(1+\epsilon)}x_i[t]\right) \le 0,$$

which implies that $\sum_{i=1}^{3}\frac{n_i[t]}{\lambda}((1+\epsilon)\lambda - \mu x_i[t]) \le 0$. Hence,

$$\sum_{i=1}^{3}\left(1 - \frac{\mu x_i[t]}{\lambda}\right) n_i[t] \le -\epsilon \sum_{i=1}^{3} n_i[t]. \tag{3}$$

Combining Eq. (1) and inequality (3), we have

$$E\left[W[t+1] - W[t]|\mathbf{n}[0]\right] \le 3 - 2\epsilon E\left[\sum_{i=1}^{3} n_i[t]\middle|\mathbf{n}[0]\right] + \sum_{i=1}^{3} E\left[\frac{\mu x_i[t]}{\lambda}\mathcal{I}_{n_i[t]>0}\middle|\mathbf{n}[0]\right]. \tag{4}$$

Then,

$$2\epsilon\cdot\frac{1}{h}\sum_{t=0}^{h-1} E\left[\sum_{i=1}^{3} n_i[t]\middle|\mathbf{n}[0]\right] \le 3 + \frac{E[W[0]|\mathbf{n}[0]]}{h} + \frac{1}{h}\sum_{t=0}^{h-1}\sum_{i=1}^{3} E\left[\frac{\mu x_i[t]}{\lambda}\mathcal{I}_{n_i[t]>0}\middle|\mathbf{n}[0]\right].$$

Taking limits on both sides, we have

$$2\epsilon \limsup_{h\to\infty}\frac{1}{h}\sum_{t=0}^{h-1} E\left[\sum_{i=1}^{3} n_i[t]\middle|\mathbf{n}[0]\right] \le 3 + \limsup_{h\to\infty}\frac{1}{h}\sum_{t=0}^{h-1}\sum_{i=1}^{3} E\left[\frac{\mu x_i[t]}{\lambda}\mathcal{I}_{n_i[t]>0}\middle|\mathbf{n}[0]\right]. \tag{5}$$

The second item can be further bound as below (see our technical report [15] for details):

$$\limsup_{h\to\infty}\frac{1}{h}\sum_{t=0}^{h-1}\sum_{i=1}^{3} E\left[\frac{\mu x_i[t]}{\lambda}\mathcal{I}_{n_i[t]>0}\middle|\mathbf{n}[0]\right] \le 3. \tag{6}$$

Combining inequalities (5) and (6), we have

$$\limsup_{h\to\infty}\frac{1}{h}\sum_{t=0}^{h-1} E\left[\sum_{i=1}^{3} n_i[t]\middle|\mathbf{n}[0]\right] \le \frac{3\rho}{1-\rho}. \tag{7}$$

Note that the left-hand side of inequality (7) is the upper limit of the long-run time average of the expected number of files in the network. According to Theorem 15.0.1 in [16] (we also give a much simpler proof in [15] for this countable state-space Markov chain), we know that since the upper limit of this long-run time average is bounded, the limit of it exists and equals to the steady-state expected number of files in the network, which gives us the desired result. □

4. Optimal resource allocation algorithm

In this section, we derive the optimal resource allocation algorithm. Specifically, we consider the problem of choosing $\mathbf{x}[t]$ as a function of $\mathbf{n}[t]$ to minimize the objective $J_h(\mathbf{n}[0])$, where

$$J_h(\mathbf{n}[0]) = E\left[\sum_{t=0}^{h}\sum_{i=1}^{3} n_i[t]\middle|\mathbf{n}[0]\right], \tag{8}$$

and the initial condition $\mathbf{n}[0]$ is assumed to be given. Thus, for a given $\mathbf{n}[0]$, we have

$$\limsup_{h\to\infty}\frac{1}{h+1}E\left[\sum_{t=0}^{h}\sum_{i=1}^{3} n_i^{OPT}[t]\middle|\mathbf{n}[0]\right] \le \limsup_{h\to\infty}\frac{1}{h+1}E\left[\sum_{t=0}^{h}\sum_{i=1}^{3} n_i^{PF}[t]\middle|\mathbf{n}[0]\right] \le \frac{3\rho}{1-\rho}.$$

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Then, according to Theorem 15.0.1 in [16], we can show that

$$E\left[\sum_{i=1}^{3} n_i^{OPT}[\infty]\right] = \min_{\mathbf{x}\in\mathcal{F}} \left\{\lim_{h\to\infty} \frac{1}{h+1} J_h(\mathbf{n}[0])\right\}.$$

Therefore, a policy which minimizes objective (8) also minimizes the steady-state expected number of files in the network, and hence is delay optimal.

4.1. Optimality of Algorithm OPT

We now begin to prove the optimality of Algorithm OPT presented in Section 2. As in the case of line networks considered in [9], we will use dynamic programming to establish the optimality of Algorithm OPT. Let

$$V_h(\mathbf{n}[0]) = \min_{\mathbf{x}\in\mathcal{F}} J_h(\mathbf{n}[0]),$$

where the minimization is over all feasible resource allocation policies.

To establish our results, we define $\boldsymbol{\delta}_1 = (1, 0, 0)$, $\boldsymbol{\delta}_2 = (0, 1, 0)$, $\boldsymbol{\delta}_3 = (0, 0, 1)$, and

$$G_h(\mathbf{n}[0]; \mathbf{x}) = \sum_{i=1}^{3} \mu x_i V_h((\mathbf{n}[0] - \boldsymbol{\delta}_i)^+) + \left(1 - \sum_{i=1}^{3}(\lambda + \mu x_i)\right) V_h(\mathbf{n}[0]). \tag{9}$$

Using dynamic programming, the cost function $V_{h+1}(\mathbf{n}[0])$ can be written as

$$V_{h+1}(\mathbf{n}[0]) = \sum_{i=1}^{3} n_i[0] + \lambda \sum_{i=1}^{3} V_h(\mathbf{n}[0] + \boldsymbol{\delta}_i) + \min_{\mathbf{x}\in\mathcal{F}} G_h(\mathbf{n}[0]; \mathbf{x}). \tag{10}$$

Lemma 4. *Given $n_{j_1}[0] \geq n_{j_2}[0]$, where $1 \leq j_1, j_2 \leq 3$ and $j_1 \neq j_2$, we have*

$$V_h((\mathbf{n}[0] - \boldsymbol{\delta}_{j_1})^+) \leq V_h((\mathbf{n}[0] - \boldsymbol{\delta}_{j_2})^+). \tag{11}$$

Proof. Without loss of generality, we assume $n_1[0] \geq n_2[0]$. First, due to the symmetry of the network, if $n_1[0] = n_2[0]$, we have $V_h((\mathbf{n}[0] - \boldsymbol{\delta}_1)^+) = V_h((\mathbf{n}[0] - \boldsymbol{\delta}_2)^+)$. Next, we use the induction to establish inequality (11) when $n_1[0] > n_2[0]$. It is easy to verify that the conclusion holds when $h = 0$. Now assume that inequality (11) holds for $h = k$, and consider $h = k + 1$:

(1) If $n_2[0] = 0$, then $V_{k+1}(\mathbf{n}[0] - \boldsymbol{\delta}_1) < V_{k+1}(\mathbf{n}[0]) = V_{k+1}((\mathbf{n}[0] - \boldsymbol{\delta}_2)^+)$.
(2) If $n_2[0] \geq 1$, then

$$\begin{aligned}
V_{k+1}(\mathbf{n}[0] - \boldsymbol{\delta}_2) &= \sum_{i=1}^{3} n_i[0] - 1 + \lambda \sum_{i=1}^{3} V_k(\mathbf{n}[0] - \delta_2 + \boldsymbol{\delta}_i) + \mu \sum_{i=1}^{3} x_i^* V_k((\mathbf{n}[0] - \boldsymbol{\delta}_2 - \boldsymbol{\delta}_i)^+) \\
&\quad + \left(1 - \sum_{i=1}^{3}(\lambda + \mu x_i^*)\right) V_k(\mathbf{n}[0] - \delta_2) \\
&\overset{(a)}{\geq} \sum_{i=1}^{3} n_i[0] - 1 + \lambda \sum_{i=1}^{3} V_k(\mathbf{n}[0] - \delta_1 + \boldsymbol{\delta}_i) + \mu \sum_{i=1}^{3} x_i^* V_k((\mathbf{n}[0] - \boldsymbol{\delta}_1 - \boldsymbol{\delta}_i)^+) \\
&\quad + \left(1 - \sum_{i=1}^{3}(\lambda + \mu x_i^*)\right) V_k(\mathbf{n}[0] - \boldsymbol{\delta}_1) \\
&\geq V_{k+1}(\mathbf{n}[0] - \boldsymbol{\delta}_1),
\end{aligned}$$

where $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{F}} G_k(\mathbf{n}[0] - \delta_2; \mathbf{x})$.

Note that the inequality (a) holds due to the induction assumption that inequality (11) holds for $h = k$, and the fact that $V_k((\mathbf{n}[0] - \boldsymbol{\delta}_1)^+) = V_k((\mathbf{n}[0] - \boldsymbol{\delta}_2)^+)$ if $n_1[0] = n_2[0]$.

Thus, we conclude that inequality (11) holds when $n_1[0] > n_2[0]$, and the lemma holds. □

Lemma 5. *Given $\mathbf{n}[0]$ such that $n_{j_1}[0] \geq n_{j_2}[0] > n_{j_3}[0] = 0$, where (j_1, j_2, j_3) is a permutation of $(1, 2, 3)$, we have*

$$(x_{j_1}^*, x_{j_2}^*, x_{j_3}^*) = (1, 0, 0) = \arg\min_{\mathbf{x}\in\mathcal{F}} G_h(\mathbf{n}[0]; \mathbf{x}).$$

Proof. Without loss of generality, we assume that $n_1[0] \geq n_2[0] > n_3[0] = 0$. We can rewrite $G_h(\mathbf{n}[0]; \mathbf{x})$ as below:

$$G_h(\mathbf{n}[0]; \mathbf{x}) = (1 - 3\lambda)a^{(h)} - \mu \sum_{i=1}^{3} x_i \Delta a_i^{(h)}, \tag{12}$$

where $a^{(h)} = V_h(\mathbf{n}[0])$ and $\Delta a_i^{(h)} = V_h(\mathbf{n}[0]) - V_h((\mathbf{n}[0]-\boldsymbol{\delta}_i)^+)$. Note that $\Delta a_i^{(h)} \geq 0$, so minimizing $G_h(\mathbf{n}[0]; \mathbf{x})$ is the same as maximizing $\sum_{i=1}^{3} x_i \Delta a_i^{(h)}$. Further note that $n_3[0] = 0$ implies that $\Delta a_3^{(h)} = 0$, so we have

$$\max_{\mathbf{x}\in\mathcal{F}} \sum_{i=1}^{3} x_i \Delta a_i^{(h)} = \max_{x_1+x_2\leq 1, x_1, x_2 > 0} \{x_1 \Delta a_1^{(h)} + x_2 \Delta a_2^{(h)}\}.$$

From Lemma 4, we have $\Delta a_1^{(h)} \geq \Delta a_2^{(h)}$ when $n_1[0] > n_2[0]$. Thus, we can conclude that

$$(1, 0) = \arg\max_{x_1+x_2\leq 1, x_1, x_2 > 0} \left\{x_1 \Delta a_1^{(h)} + x_2 \Delta a_2^{(h)}\right\}.$$

Besides, when $n_1[0] = n_2[0]$, due to symmetry we know that $\Delta a_1^{(h)} = \Delta a_2^{(h)}$, so serving either file type 1 or 2 at rate 1 is optimal. Obviously, $x_3^* = 0$. Therefore the lemma holds. □

Lemma 6. *Given* $\mathbf{n}[0]$ *such that* $n_1[0], n_2[0], n_3[0] > 0$, *we have*

$$(0.5, 0.5, 0.5) = \arg\min_{\mathbf{x}\in\mathcal{F}} G_h(\mathbf{n}[0]; \mathbf{x}).$$

Proof. Without loss of generality, we assume that

$$n_1[0] \geq n_2[0] \geq n_3[0] > 0. \tag{13}$$

We further assume that $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = \arg\min_{\mathbf{x}\in\mathcal{F}} G_h(\mathbf{n}[0]; \mathbf{x})$. It can be shown that $x_1^* \geq 0.5$. (Details are in our technical report [15].) We then want to show that $x_1^* \leq 0.5$. First, we have

$$\begin{aligned} 0 &\geq G_h(\mathbf{n}[0]; \mathbf{x}^*) - G_h(\mathbf{n}[0]; \mathbf{0.5}) \\ &= (x_1^* - 0.5)\cdot \mu(V_h(\mathbf{n}[0]) + V_h(\mathbf{n}[0]-\delta_1) - V_h(\mathbf{n}[0]-\delta_2) - V_h(\mathbf{n}[0]-\delta_3)), \end{aligned} \tag{14}$$

where $\mathbf{0.5} = (0.5, 0.5, 0.5)$. Thus, under the assumption $n_1[0] \geq n_2[0] \geq n_3[0] > 0$ and the fact $x_1^* \geq 0.5$, it is easy to show that $\mathbf{x}^* = \mathbf{0.5}$ holds if and only if the following inequality holds:

$$V_h(\mathbf{n}[0]) + V_h(\mathbf{n}[0]-\delta_1) > V_h(\mathbf{n}[0]-\delta_2) + V_h(\mathbf{n}[0]-\delta_3). \tag{15}$$

Therefore, we focus on inequality (15) in the following analysis. First consider $n_1[0] = n_2[0]$. Due to symmetry, it is easy to see that $V_h(\mathbf{n}[0]-\delta_1) = V_h(\mathbf{n}[0]-\delta_2)$. Also it is obvious to see that $V_h(\mathbf{n}[0]) > V_h(\mathbf{n}[0]-\delta_3)$. Thus, inequality (15) holds for all $h \geq 0$ when $n_1[0] = n_2[0]$.

If $n_1[0] > n_2[0]$, which implies that $n_1[0] \geq 2$, we can use the induction to establish inequality (15). It is easy to verify that it holds for $h = 0$. We assume that it holds for $h = k$, and then consider $h = k + 1$. Recall that $3\lambda + \frac{3\mu}{2} = 1$. Therefore, we have

$$V_{k+1}(\mathbf{n}[0]) = \sum_i n_i[0] + \lambda \sum_{i=1}^{3} V_k(\mathbf{n}[0] + \boldsymbol{\delta}_i) + \frac{\mu}{2} \sum_{i=1}^{3} V_k(\mathbf{n}[0] - \boldsymbol{\delta}_i).$$

Similarly, since $n_1[0] - 1 \geq n_2[0] \geq n_3[0]$, we also have

$$V_{k+1}(\mathbf{n}[0]-\delta_1) = \sum_i n_i[0] - 1 + \lambda \sum_{i=1}^{3} V_k(\mathbf{n}[0] - \delta_1 + \boldsymbol{\delta}_i) + \frac{\mu}{2} \sum_{i=1}^{3} V_k(\mathbf{n}[0] - \delta_1 - \boldsymbol{\delta}_i).$$

On the other hand, we have

$$\begin{aligned} &V_{k+1}(\mathbf{n}[0]-\delta_2) + V_{k+1}(\mathbf{n}[0]-\delta_3) \\ &\overset{(a)}{\leq} 2\sum_i n_i[0] - 2 + \lambda \sum_{i=1}^{3} (V_k(\mathbf{n}[0]-\delta_2+\delta_i) + V_k(\mathbf{n}[0]-\delta_3+\delta_i)) + G_k(\mathbf{n}[0]-\delta_2; \delta_1) + G_k(\mathbf{n}[0]-\delta_3; \delta_1) \\ &\overset{(b)}{<} 2\sum_i n_i[0] - 1 + \lambda \sum_{i=1}^{3} (V_k(\mathbf{n}[0]-\delta_1+\delta_i) + V_k(\mathbf{n}[0]+\delta_i)) + \frac{\mu}{2} \sum_{i=1}^{3} (V_k(\mathbf{n}[0]-\delta_1-\delta_i) + V_k(\mathbf{n}[0]-\delta_i)) \\ &= V_{k+1}(\mathbf{n}[0]-\delta_1) + V_{k+1}(\mathbf{n}[0]). \end{aligned}$$

Note that inequality (a) holds since $(1, 0, 0) \in \mathcal{F}$, but may not be the optimal resource allocation at time slot k. Inequality (b) holds since

$$V_k(\mathbf{n}[0]-\delta_1-\delta_2) + V_k(\mathbf{n}[0]-\delta_1-\delta_3) < V_k(\mathbf{n}[0]-\delta_1-\delta_1) + V_k(\mathbf{n}[0]-\delta_1)$$

holds for k when $n_1[0]-1 > n_2[0] \geq n_3[0]$ (the induction assumption), and holds for any $h \geq 0$ if $n_1[0]-1 = n_2[0] \geq n_3[0]$. Therefore the lemma holds. □

Theorem 7. *For the three-link, symmetric star network with $\rho < 1$, given $\mathbf{n}[0]$, Algorithm OPT minimizes the objective function $J_h(\mathbf{n}[0])$ defined in (8).*

Proof. The proof directly follows from Lemmas 6, 5, and an obvious fact that the optimal policy should serve the only nonempty file type at rate 1 if only one file type is nonempty. Details can be found in our technical report [15]. □

4.2. Lower bound on steady-state expected number of files

In this subsection, we prove Theorem 3, i.e., the lower bound on $E\left[\sum_{i=1}^{3} n_i^{OPT}[\infty]\right]$.

Proof of Theorem 3. We consider the evolution of $\mathbf{n}[t]$ under the optimal algorithm. We ignore the superscript without causing any confusion. According to the optimal policy, the file types with the largest and the second largest numbers of files always get a combined service rate of 1 (if nonempty), while the file type with the smallest number of files always gets a service rate of 0.5 (if nonempty). Hence, in our proof, we will not directly use $n_i[t]$ but instead use $m_i[t]$ defined below:

$$m_1[t] = \max_{i=1,2,3} n_i[t], \qquad m_3[t] = \min_{i=1,2,3} n_i[t], \qquad m_2[t] = \{n_1[t], n_2[t], n_3[t]\} \setminus \{m_1[t], m_3[t]\}.$$

We construct a Lyapunov function which is a sum of two separate quadratic forms as follows:

$$W[t] = (m_1[t] + m_2[t])^2 + k\,(m_3[t])^2\,, \tag{16}$$

where k is a constant which will be chosen later to get a lower bound on the expected number of files in the network. It can be shown that (see our technical report [15] for details)

$$\begin{aligned} E\left[W[t+1] - W[t] | \mathbf{m}[t]\right] \geq{} & 2\lambda\left(2(m_1[t] + m_2[t]) + 1\right) + \lambda\left(2k \cdot m_3[t] + k\right) + \mu \mathcal{I}_{m_1[t]>0}\left(-2\left(m_1[t] + m_2[t]\right) + 1\right) \\ & + \frac{\mu \mathcal{I}_{m_3[t]>0}}{2}\left(-2k \cdot m_3[t] + k\right) - (k-1)\lambda \mathcal{I}_{m_1[t]=0}. \end{aligned} \tag{17}$$

In order to obtain the term $E[\sum_{i=1}^{3} m_i[t]]$ in the above inequality, it is clear that we should let $k = 2$. Thus, continuing from inequality (17), we have

$$\begin{aligned} E\left[W[t+1] - W[t] | \mathbf{m}[t]\right] &\geq 2\sum_{i=1}^{3} m_i[t] \cdot (2\lambda - \mu) + 4\lambda + \frac{\mu}{2}\left(2\mathcal{I}_{m_1[t]>0} + 2\mathcal{I}_{m_3[t]>0}\right) - \lambda \mathcal{I}_{m_1[t]=0} \\ &= 2\sum_{i=1}^{3} m_i[t] \cdot (2\lambda - \mu) + 4\lambda + \frac{\mu}{2}\left(2\mathcal{I}_{m_1[t]>0} + \mathcal{I}_{m_3[t]>0}\right) + \frac{\mu}{2}\mathcal{I}_{m_3[t]>0} - \lambda \mathcal{I}_{m_1[t]=0}. \end{aligned} \tag{18}$$

Hence, we have

$$\begin{aligned} \frac{1}{h}\sum_{t=0}^{h-1} E\left[W[t+1] - W[t] | \mathbf{n}[0]\right] \geq{} & (4\lambda - 2\mu)\frac{1}{h}\sum_{t=0}^{h-1} E\left[\sum_{i=1}^{3} m_i[t] \middle| \mathbf{n}[0]\right] + 4\lambda \\ & + \frac{\mu}{2}\left(2 \cdot \frac{1}{h}\sum_{t=0}^{h-1} \Pr(m_1[t] > 0|\mathbf{n}[0]) + \frac{1}{h}\sum_{t=0}^{h-1} \Pr(m_3[t] > 0|\mathbf{n}[0])\right) \\ & + \frac{\mu}{2} \cdot \frac{1}{h}\sum_{t=0}^{h-1} \Pr(m_3[t] > 0|\mathbf{n}[0]) - \lambda \cdot \frac{1}{h}\sum_{t=0}^{h-1} \Pr(m_1[t] = 0|\mathbf{n}[0]). \end{aligned} \tag{19}$$

According to Theorem 15.0.1 in [16], inequality (19) can be further written as

$$\begin{aligned} \limsup_{h\to\infty} \frac{1}{h} E\left[W[h] | \mathbf{n}[0]\right] \geq{} & (4\lambda - 2\mu) E\left[\sum_{i=1}^{3} n_i[\infty]\right] + 4\lambda + \frac{\mu}{2}\left(2\Pr(m_1[\infty] > 0) + \Pr(m_3[\infty] > 0)\right) \\ & + \frac{\mu}{2}\Pr(m_3[\infty] > 0) - \lambda \Pr(m_1[\infty] = 0). \end{aligned} \tag{20}$$

Furthermore, we note that $E\left[\frac{1}{2}\left(2\mathcal{I}_{m_1[\infty]>0} + \mathcal{I}_{m_3[\infty]>0}\right)\right]$ is the average service rate the network receives at the steady state. Intuitively, it should be equal to the total arrival rate, i.e.,

$$E\left[\frac{\mu}{2}\left(2\mathcal{I}_{m_1[\infty]>0} + \mathcal{I}_{m_3[\infty]>0}\right)\right] = 3\lambda. \tag{21}$$

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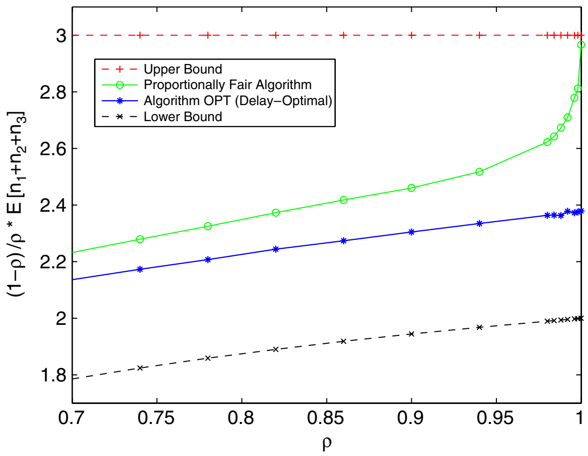

Fig. 2. Scaled long-run average number of the files in the network.

It can also be shown that

$$\limsup_{h\to\infty} \frac{1}{h} E\left[W[h] | \mathbf{n}[0]\right] = 0. \tag{22}$$

The detailed proofs for inequality (20) and Eqs. (21) and (22) can be found in our technical report [15]. Combining inequality (20) and Eqs. (21) and (22), we can conclude that

$$E\left[\sum_{i=1}^{3} n_i[\infty]\right] \geq \frac{7\lambda + \frac{\mu}{2}P_2 - \lambda P_0}{2(\mu - 2\lambda)} = \frac{7\rho + P_2 - \rho P_0}{4(1-\rho)},$$

where $P_0 \triangleq \Pr(m_1[\infty] = 0) = \Pr(n_1[\infty] = n_2[\infty] = n_3[\infty] = 0)$ and $P_2 \triangleq \Pr(m_3[\infty] > 0) = \Pr(n_i[\infty] > 0, \forall i)$. Recall that $\rho = 2\lambda/\mu$. We further define that $P_1 \triangleq \Pr(\text{one or two file types are empty})$. Then the following equations hold:

$$\begin{cases} P_0 + P_1 + P_2 = 1 \\ \dfrac{3}{2}\mu P_2 + \mu P_1 = 3\lambda, \end{cases}$$

where the second equation is obtained from Eq. (21). Then, we get $P_0 = 1 - \frac{3}{2}\rho + \frac{1}{2}P_2$. Since $P_0 \geq 0$, we get $P_2 \geq 3\rho - 2$. Therefore, we have

$$E\left[\sum_{i=1}^{3} n_i[\infty]\right] \geq \frac{\frac{3}{2}\rho^2 + 6\rho + (1-\frac{\rho}{2})P_2}{4(1-\rho)} \geq \frac{\frac{3}{2}\rho^2 + 6\rho + (1-\frac{\rho}{2})(3\rho-2)}{4(1-\rho)} = \frac{5\rho - 1}{2(1-\rho)}. \quad \square$$

4.3. Simulation results

In Fig. 2, we have plotted the "scaled long-run average number of files in the network", i.e., $\left(\frac{1-\rho}{\rho}\right) \cdot E[n_1 + n_2 + n_3]$, obtained through a simulation, respectively for proportionally fair algorithm (details of the algorithm description and how to derive it can be found in our technical report [15]) and the optimal algorithm, and compared them with the bounds that we have proved in the earlier sections. From the figure, we note the following facts:

- The curve for the scaled number of files under the proportionally fair algorithm is convex and gets close to the upper bound as $\rho \to 1$. The figure suggests that the upper bound is tight when $\rho = 1$.
- The curve for the scaled number of files under the optimal algorithm is concave and is almost parallel to the curve of the lower bound, i.e., $(5\rho - 1)/2\rho$. As $\rho \to 1$, the lower bound approaches 2, while the optimal algorithm reaches a limit of around 2.4. This implies that the lower bound is not tight.
- From the two simulation curves, the gain of the optimal algorithm over the proportionally fair algorithm increases with the workload on each link (ρ). In the heavy traffic regime ($\rho \approx 1$), the gain reaches about 1.25.

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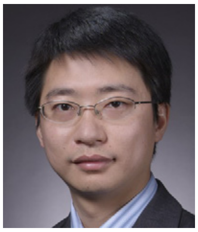

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